Universal Types

Motivation

In the simply typed lambda-calculus, we often have to write several versions of the same code, differing only in type annotations.

```
doubleNat = λf:Nat→Nat. λx:Nat. f (f x)
doubleRcd = λf:{l:Bool}→{l:Bool}. λx:{l:Bool}. f (f x)
doubleFun = λf:(Nat→Nat)→(Nat→Nat). λx:Nat→Nat. f (f x)
```

Bad! Violates a basic principle of software engineering:

> Write each piece of functionality once

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Here, the details that vary are the types!

Idea

We'd like to be able to take a piece of code and "abstract out" some type annotations.

We've already got a mechanism for doing this with terms: λ-abstraction. So let's just re-use the notation.

Abstraction:

```
double = λX. λf:X→X. λx:X. f (f x)
```

Application:

```
double [Nat]
double [Bool]
```

Computation:

```
double [Nat] ⟶ λf:Nat→Nat. λx:Nat. f (f x)
```

(N.b.: Type application is commonly written `t [T]`, though `t T` would be more consistent.)

Idea

What is the *type* of a term like

λ`X. `λ`f:X`$\rightarrow$`X. `λ`x:X. f (f x)` ?

This term is a function that, when applied to a type `X`, yields a term of type `(X`$\rightarrow$`X)`$\rightarrow$`X`$\rightarrow$`X`.

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$$\lambda\texttt{X. }\lambda\texttt{f:X}\rightarrow\texttt{X. }\lambda\texttt{x:X. f (f x)}\ ?$$

This term is a function that, when applied to a type $\texttt{X}$, yields a term of type $\texttt{(X}\rightarrow\texttt{X)}\rightarrow\texttt{X}\rightarrow\texttt{X}$.

I.e., for all types $\texttt{X}$, it yields a result of type $\texttt{(X}\rightarrow\texttt{X)}\rightarrow\texttt{X}\rightarrow\texttt{X}$.

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We'll write it like this: $\forall\texttt{X}.\ \texttt{(X}\rightarrow\texttt{X)}\rightarrow\texttt{X}\rightarrow\texttt{X}$

System F

System F (aka "the polymorphic lambda-calculus") formalizes this idea by extending the simply typed lambda-calculus with type abstraction and type application.

`t ::=`		*terms*
	`x`	*variable*
	λ`x:T.t`	*abstraction*
	`t t`	*application*
	λ`X.t`	*type abstraction*
	`t [T]`	*type application*

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	$\texttt{t t}$	*application*
	$\lambda\texttt{X.t}$	*type abstraction*
	$\texttt{t [T]}$	*type application*
$\texttt{v}$ ::=		*values*
	$\lambda\texttt{x:T.t}$	*abstraction value*
	$\lambda\texttt{X.t}$	*type abstraction value*

System F: new evaluation rules

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\ [\mathtt{T}_2] \longrightarrow \mathtt{t}_1'\ [\mathtt{T}_2]} \qquad \text{(E-TApp)}$$

$$(\lambda \mathtt{X}.\mathtt{t}_{12})\ [\mathtt{T}_2] \longrightarrow [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{t}_{12} \qquad \text{(E-TappTabs)}$$

System F: Types

To talk about the types of "terms abstracted on types," we need to introduce a new form of types:

T ::=		*types*
	X	*type variable*
	$T \rightarrow T$	*type of functions*
	$\forall X.T$	*universal type*

System F: Typing Rules

$$\frac{\mathtt{x}:\mathtt{T} \in \Gamma}{\Gamma \vdash \mathtt{x} : \mathtt{T}} \qquad \text{(T-Var)}$$

$$\frac{\Gamma, \mathtt{x}:\mathtt{T_1} \vdash \mathtt{t_2} : \mathtt{T_2}}{\Gamma \vdash \lambda \mathtt{x}:\mathtt{T_1}.\mathtt{t_2} : \mathtt{T_1} \to \mathtt{T_2}} \qquad \text{(T-Abs)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \mathtt{T_{11}} \to \mathtt{T_{12}} \qquad \Gamma \vdash \mathtt{t_2} : \mathtt{T_{11}}}{\Gamma \vdash \mathtt{t_1}\ \mathtt{t_2} : \mathtt{T_{12}}} \qquad \text{(T-App)}$$

$$\frac{\Gamma, \mathtt{X} \vdash \mathtt{t_2} : \mathtt{T_2}}{\Gamma \vdash \lambda \mathtt{X}.\mathtt{t_2} : \forall \mathtt{X}.\mathtt{T_2}} \qquad \text{(T-TAbs)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \forall \mathtt{X}.\mathtt{T_{12}}}{\Gamma \vdash \mathtt{t_1}\ [\mathtt{T_2}] : [\mathtt{X} \mapsto \mathtt{T_2}]\mathtt{T_{12}}} \qquad \text{(T-TApp)}$$

History

Interestingly, System F was invented independently and almost simultaneously by a computer scientist (John Reynolds) and a logician (Jean-Yves Girard).

Their results look very different at first sight — one is presented as a tiny programming language, the other as a variety of second-order logic.

The similarity (indeed, isomorphism!) between them is an example of the *Curry-Howard Correspondence*.